Math 255B Lecture 26 Notes

Daniel Raban

March 11, 2020

1 Weyl's Criterion and Weyl's Theorem

1.1 Weyl's criterion

Last time, we had $\mathrm{Spec}(S) = \mathrm{Spec}_d(A) \sqcup \mathrm{Spec}_{\mathrm{ess}}(A)$, where $\mathrm{Spec}_d(A)$ is the set of isolated eigenvalues of A of finite multiplicity.

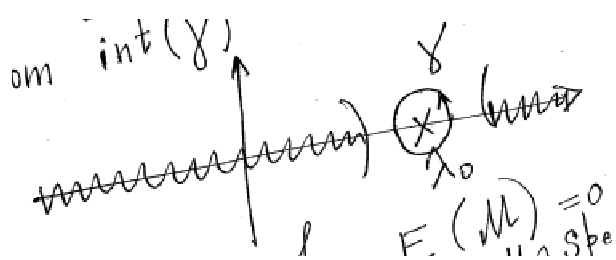
Proposition 1.1 (Weyl's criterion)**.** *$\lambda \in \mathrm{Spec}_{\mathrm{ess}}(A)$ if and only if there exists a sequence $u_n \in D(A)$ with $\|u_n\| = 1$, such that $u_n \to 0$ weakly and $\|(A - \lambda)u_n\| \to 0$.*

Such a sequence is called a **Weyl sequence**.

Lemma 1.1. *Let λ be an isolated point of* $\mathrm{Spec}(A)$*. Then λ_0 is an eigenvalue, and*

$$E(\{\lambda_0\}) = \frac{1}{2\pi i}\int_\gamma (z - A)^{-1}\,dz,$$

where γ is a small circle centered at λ_0 with $\mathrm{Spec}(A) \setminus \{\lambda\}$ *away from* $\overline{\mathrm{int}(\gamma)}$*.*



Proof. $E(M) = 0$ if $M \cap \mathrm{Spec}(A) = \varnothing$. For all small enough $\varepsilon > 0$, we have

$$0 \neq E((\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)) = E(\{\lambda_0\}),$$

so λ_0 is an eigenvalue. Compute

$$\frac{1}{2\pi i}\int_\gamma \langle (z - A)^{-1}u, u\rangle\,dz = \frac{1}{2\pi i}\int_\gamma \int \frac{1}{z - \lambda}\,d\langle E_\lambda u, u\rangle\,dz$$

$$= \int \left(\frac{1}{2\pi i} \int_\gamma \frac{1}{z-\lambda}\, dz \right) d\langle E_\lambda u, u\rangle$$

The part in the parentheses equals 1 if $\lambda \in \operatorname{int}(\gamma)$ and 0 if $\lambda \notin \overline{\operatorname{int}(\gamma)}$.

$$\begin{aligned} &= \int \mathbb{1}_{\operatorname{int}(\gamma)\cap\mathbb{R}}(\lambda)\, d\langle E_\lambda u, u\rangle \\ &= \langle E(\{\lambda_0\})u, u\rangle \\ &= \langle E(\operatorname{int}(\gamma))u, u\rangle. \end{aligned}$$

□

Now we can prove Weyl's criterion.

Proof. ($\Longleftarrow$): Assume that there is a sequence $u_n \in D(A)$ with $\|u_n\| = 1$, such that $u_n \to 0$ weakly and $\|(A-\lambda)u_n\| \to 0$. Then $\lambda \in \operatorname{Spec}(A)$. Assume that $\lambda \in \operatorname{Spec}_d(A)$. Then, using

$$(z-A)u_n = (\lambda - A)u_n) + (z-\lambda)u_n,$$

we have

$$(z-A)^{-1}u_n = \frac{1}{z-\lambda}u_n - (z-A)^{-1}\frac{1}{z-\lambda}(\lambda - A)u_n.$$

Integrate over γ to get

$$E(\{\lambda\})u_n = u_n - \underbrace{\frac{1}{2\pi i}\int_\gamma (z-A)^{-1}(z-\lambda)^{-1}(\lambda-A)u_n\, dz}_{\to 0 \text{ in } \mathcal{H}}.$$

We get that $\|E(\{\lambda\})u_n - u_n\| \to 0$. $E(\{\lambda\})$ is finite rank, so it is compact. Therefore, $E(\{\lambda\})u_n \to 0$, which contradicts the fact that $\|u_n\| = 1$.

($\Longrightarrow$): Let $\lambda \in \operatorname{Spec}_{\text{ess}}(A)$. We claim that for all $\varepsilon > 0$, $\dim E((\lambda-\varepsilon, \lambda+\varepsilon))\mathcal{H} = \infty$. Indeed, assume that $\dim E(I)\mathcal{H} < \infty$ for some $I = (\lambda - \varepsilon, \lambda+\varepsilon)$. Then write

$$\mathcal{H} = E(I)\mathcal{H} \oplus E(I^c)\mathcal{H},$$

which is a closed, orthogonal direct sum. This composition reduces A, and $\operatorname{Spec}(A|_{E(I)\mathcal{H}}) \cap I \neq \varnothing$. Hence, $\operatorname{Spec}(A) \cap I = \operatorname{Spec}(A|_{E(I)\mathcal{H}})$, which is finite, of finite multiplicity. So $\lambda \in \operatorname{Spec}_d(A)$.

If $\dim E(\{\lambda\})\mathcal{H} = \infty$ (λ is an eigenvalue of ∞ multiplicity), we let $u_n \in E(\{\lambda\})\mathcal{H}$ be an orthonormal sequence. In general, we can find a sequence $\varepsilon_n \downarrow 0$ such that

$$P_n = E((\lambda - \varepsilon_n, \lambda - \varepsilon_{n+1}) \cup (\lambda + \varepsilon_{n+1}, \lambda - \varepsilon_n)) \neq 0, \qquad P_n P_m = 0, \quad n \neq m.$$

It suffices to take $u_n \in P_n\mathcal{H}$ of norm 1. □

1.2 Weyl's theorem

Theorem 1.1 (Weyl)**.** *Let A, B be self-adjoint, bounded from below, and such that for some $c \in \mathbb{R}$, $(A+c)^{-1} - (B+c)^{-1}$ is compact. Then* $\mathrm{Spec}_{\mathrm{ess}}(A) = \mathrm{Spec}_{\mathrm{ess}}(B)$.

Proof. We check that $\mathrm{Spec}_{\mathrm{ess}}(A) \subseteq \mathrm{Spec}_{\mathrm{ess}}(B)$. Let $\lambda \in \mathrm{Spec}_{\mathrm{ess}}(A)$, let u_n be a Weyl sequence for A at λ, and let $v_n = (B+c)^{-1}(A+c)u_n \in D(B)$. Write $(B+c)^{-1} = (A+c)^{-1} + K$, where K is compact. Then

$$\begin{aligned} v_n &= u_n + K(A+c)u_n \\ &= u_n + \underbrace{K(A-\lambda)u_n}_{\to 0} + \underbrace{K\underbrace{(\lambda+c)u_n}_{\to 0 \text{ weakly}}}_{\to 0}. \end{aligned}$$

So $\|v_n\| \geq 1/2$ for all large n. Hence, $w_n = \frac{v_n}{\|v_n\|} \in D(B)$, $w_n \to 0$ weakly, and $(B-\lambda)w_n \to 0$ since

$$\begin{aligned} (B-\lambda)v_n) &= (B+c)v_n - (-\lambda - c)v_n \\ &= (A+c)u_n - (\lambda+c)v_n \\ &= \underbrace{(A-\lambda)u_n}_{\to 0} - (\lambda+c)\underbrace{(v_n - u_n)}_{\to 0}. \end{aligned}$$

□

1.3 Applications

Example 1.1. Consider $P_0 = -\Delta$ on $\mathbb{R}^n$, and let $P = P_0 + q$ with $q \in C(\mathbb{R}^n; \mathbb{R})$ such that $q \to 0$ as $|x| \to \infty$. P and P_0 are self-adjoint with $D(P) = D(P_0) = H^2(\mathbb{R}^n)$, and let us check that $(P+c)^{-1} - (P_0+c)^{-1}$ is compact:

Let us write $(P+x)u = (P_0+c)u + qu$ and replace $u \in H^2$ by $(P_0+c)^{-1}u$ for $u \in H^2(\mathbb{R}^n)$. Then $u \in L^2$ with

$$(P+c)(P_0+c)^{-1}u = u + q(P_0+c)^{-1}u.$$

Apply $(P+c)^{-1}$ to get

$$(P_0+c)^{-1}u = (P+c)^{-1}u + (P+c)^{-1}q(P_0+c)^{-1}u.$$

We get the resolvent identity

$$(P+c)^{-1} - (P_0+c)^{-1} = -(P+c)^{-1}q(P_0+c)^{-1}.$$

Let us check that $q(P_0+c)^{-1} : L^2 \to L^2$ is compact: Approximating q by a sequence $q_j \in \overline{C_0}$ (uniformly), we may assume that $q \in C_0(\mathbb{R})$. If $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi q = q$, we get

$$q(P_0+c)^{-1} = q(\chi(P_0+c)^{-1}) : L^2 \to H^2 \cap \mathcal{E}(\mathrm{supp}(\chi)) \to L^2,$$

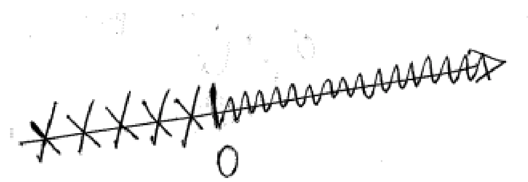
where the second map is compact. We get

$$\mathrm{Spec}_{\mathrm{ess}}(P) = \mathrm{Spec}_{\mathrm{ess}}(P_0) = [-\infty),$$

so

$$\mathrm{Spec}(P) = [0, \infty) \cup \{\text{negative eigenvalues}\},$$

where negative eigenvalues may only accumulate at 0.



If q is rapidly decreasing at ∞, then $\mathrm{Spec}(P) \cap (-\infty)$ is finite and

$$\#\text{ of negative eigenvalues} \leq C_n \int |q|^{n/2}\, dx.$$

For $n \geq 3$, this is the Cwikel-Lieb-Rozenblum estimate (1972-1977). Observe that there are no negative eigenvalues if q is small.

Example 1.2. Let $P = -\Delta + q$ with $q \in C(\mathbb{R}^n)$ and $q \geq 0$. Let $\eta = \liminf_{|x| \to \infty} q(x) \in [0, \infty]$. We claim that $\inf \mathrm{Spec}_{\mathrm{ess}}(P) \geq \eta$ ($\mathrm{Spec}(P)$ is *discrete* in $(0, \eta)$). We may assume that $\eta < \infty$. Let

$$A = -\Delta + \max(q, \eta), \qquad B = q - \max(q, \eta) \in C(\mathbb{R}^n).$$

Then $B(x) \to 0$ as $|x| \to +\infty$. We get that $B(A + c)^{-1}$ is compact on L^2, since $(A + c)^{-1}(L^2) \subseteq H^1(\mathbb{R}^n)$. Thus,

$$\mathrm{Spec}_{\mathrm{ess}}(\underbrace{-\Delta + q}_{=A+B}) = \mathrm{Spec}_{\mathrm{ess}}(A) \subseteq \mathrm{Spec}(A) \subseteq [\eta, \infty).$$